Linearized Richtmyer-Meshkov analysis for impulsively accelerated elastic solids

Alejandro Lopez-Ortega
David Hill, Dale Pullin, Dan Meiron

Computational Fluid Dynamics Group
Caltech PSAAP Center

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Overview

- RM impulsive instability of elastic solid
  - Neo-Hookean elastic solid
  - 2D with plain strain
- Eulerian approach to solid continuum equations
- Initial-value problem
  - Time-domain solution using Laplace transform
  - Single-mode interfacial initial perturbation
  - Inversion using Bromwich contour
- Interface behavior as a function of material properties
- Vorticity distribution
- Previous semi-analytical studies:
  - Plohr and Plohr (2005)
  - Piriz et. Al (2006).

Impulsive Richtmyer-Meshkov

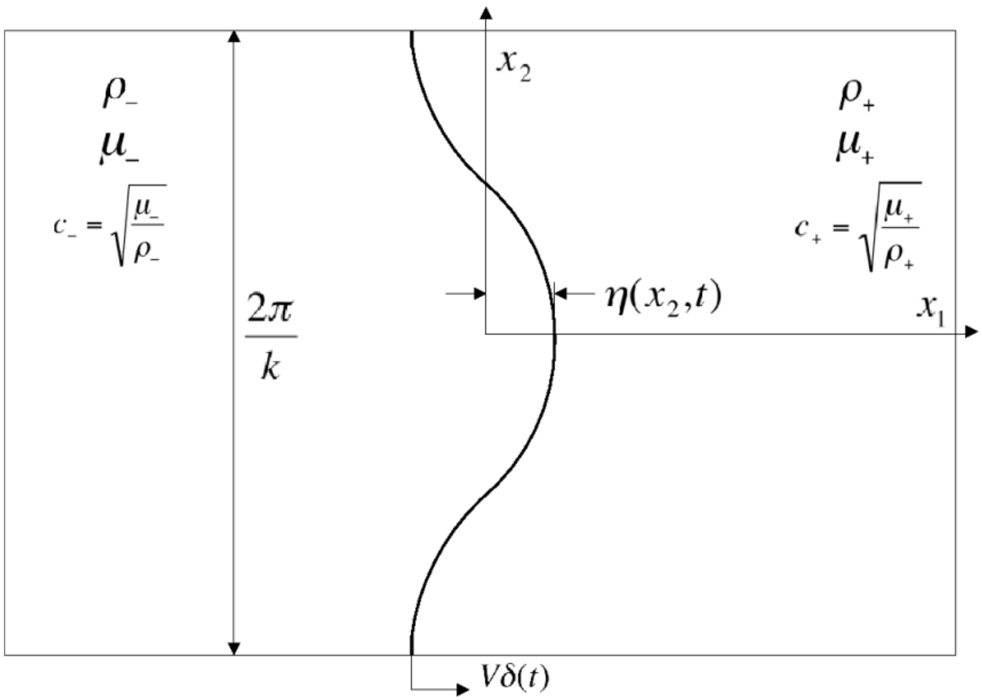
Gas dynamics, Richtmyer '60, interface grows linearly in time

Simplifying assumptions:

- Elastic, incompressible, Neo-Hookean
- Shockwave replaced by impulse
- Small interfacial perturbation

Advantages:

- Analytically tractable
- Comprehensive exploration of parameter space (relative densities and shear speeds)



$c = \sqrt{\frac{\mu}{\rho}}$: Shear-wave speed

μ : shear modulus

Eulerian equations of Motion

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Lagrangian map:

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$$

$\boldsymbol{x}$:position of particle originally at $\boldsymbol{X}$

$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

Deformation tensor

Inverse map:

$$\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t)$$

$$g_{ij} = \frac{\partial X_i}{\partial x_j}, \qquad F_{ij}\, g_{jk} = \delta_{ik}, \qquad : g_{11}\, g_{22} - g_{21}\, g_{22} = 1$$

Inverse deformation tensor

Eulerian equations of motion

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial \sigma_{ij}}{\partial x_j} = 0,$$

$$\frac{\partial g_{ij}}{\partial t} + \frac{\partial}{\partial x_j}(g_{ik} u_k) = 0.$$

Neo-Hookean elastic solid

$$\sigma_{ij} = -p\delta_{ij} + \mu F_{ik} F_{jk}$$

$$\frac{\partial^2 p}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j}(-\rho u_i u_j + \mu F_{ik} F_{jk}),$$

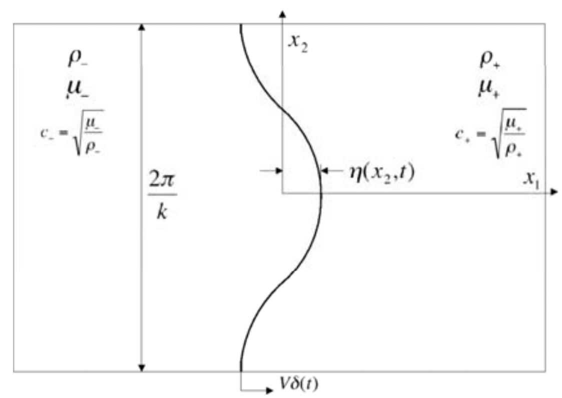
Plane strain assumed

Initial and interface conditions

Interfacial conditions:

$$[|u_i|]n_i = 0$$
$$n_i[|\sigma_{ij}|]n_j = 0$$
$$t_i\sigma^-_{ij}n_j = 0 = t_i\sigma^+_{ij}n_j$$

Continuity of normal velocity
Continuity of normal stress
Zero tangential stress



Interfacial kinematics:

$$\frac{D}{Dt}\left(\eta(x_2,t) - x_1\right) = 0$$

Base flow:

$$u_1 = -V\left(H(t) - 1\right), \qquad V = \text{Impulse velocity}$$
$$u_2 = 0$$
$$g_{ij} = \delta_{ij}$$
$$p = \rho V\delta(t)x + \mu, \quad \sigma_{ij} = -\rho V\delta(t)x\,\delta_{ij}$$

Linearized equations

Linearization:

$$\frac{\partial u_j'}{\partial x_j} = 0$$

$$\rho \frac{\partial u_j'}{\partial t} + \overline{u_1} \frac{\partial u_i'}{\partial x_1} - \frac{\partial \sigma_{ij}'}{\partial x_i} = 0$$

$$\frac{\partial \eta'}{\partial t} - u_1' = 0$$

$$\frac{\partial g_{ij}'}{\partial t} + \overline{u_1} \frac{\partial g_{i1}'}{\partial x_j} + \frac{\partial u_i'}{\partial x_j} = 0$$

$$\sigma_{ij}' = -p'\delta_{ij} - \mu(g_{ij}' + g_{ji}')$$

Linearized interface conditions:

$$\begin{aligned} [|u_1'|] &= 0 \\ [|\sigma_{11}'|] &= -[|\rho|]\, V \eta' \delta(t) \\ \sigma_{21}'^{-} = \ & 0 = \sigma_{21}'^{+} \end{aligned}$$

Laplace transform

Solution:

$$q'(t, x_1, x_2) \quad = \quad \hat{q}(x_1, t)\exp(ikx_2)$$

Sinusoidal in x_2:

$$Q(x_1, s) \quad \equiv \quad \mathcal{L}[\hat{q}(x_1, t)] = \int_0^\infty \hat{g}_{ij}(t)\, e^{-s\,t}\, dt$$

Laplace transform in t

Linearized equations:

$$\frac{d}{dx}U_1 + ikU_2 = 0$$

$$\rho s U_1 + 2\mu \frac{d}{dx_1}G_{11} + \frac{d}{dx_1}P + \mu ik(G_{12} + G_{21}) = 0,$$

$$\rho s U_2 + 2\mu ik G_{22} + ikP + \mu \frac{d}{dx_1}(G_{12} + G_{21}) = 0$$

$$sG_{i1} + \frac{d}{dx_1}U_i = 0$$

$$sG_{i2} + ikU_i = 0$$

Solution in s-plane

4-th order ODE for U_1

$$\frac{d^4U_1}{dx_1^4} - \left(\frac{s^2}{c^2} - 2k^2\right)\frac{d^2U_1}{dx_1^2} + \left(\frac{k^2s^2}{c^2} + k^4\right)U_1 = 0$$

$$U_1^{\pm}(s,x_1) = A_{\pm}(s)e^{\mp kx_1} + B_{\pm}(s)e^{\mp\sqrt{\frac{s^2}{c_{\pm}^2}+k^2}x_1}$$

$$A_{\pm}(s) = \frac{(r-1)V\eta_0 ks(2c_{\pm}^2k^2+s^2)}{\Omega(s)}, \qquad r = \frac{\rho_+}{\rho_1} = \frac{1+A}{1-A}$$

$$B_{\pm}(s) = -\frac{(r-1)V\eta_0 ks2c_{\pm}^2k^2}{\Omega(s)}$$

$$\Omega(s) = (1+r)s^4 + 4c_-^2k^2s^2 + 4c_+^2k^2rs^2 - 4c_-^4k^4\left(-1+\sqrt{1+\frac{s^2}{c_-^2k^2}}\right) - 4c_+^4k^4r\left(-1+\sqrt{1+\frac{s^2}{c_+^2k^2}}\right)$$

Solution in s-plane

4-th order ODE for U_1

$$\frac{d^4U_1}{dx_1^4} - \left(\frac{s^2}{c^2} - 2k^2\right)\frac{d^2U_1}{dx_1^2} + \left(\frac{k^2s^2}{c^2} + k^4\right)U_1 = 0$$

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Inverse Laplace transform

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$$\hat{u}_1(x_1,t) = \mathcal{L}^{-1}\{U_1(s)\} \equiv \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}U_1(x_1,s)\,ds$$

Detailed solution depends on singularities of the denominator:

$$\begin{aligned}\Omega(s) &= (1+r)s^4 + 4c_-^2k^2s^2 + 4c_+^2k^2rs^2 \\ &- 4c_-^4k^4\left(-1+\sqrt{1+\frac{s^2}{c_-^2k^2}}\right) - 4c_+^4k^4r\left(-1+\sqrt{1+\frac{s^2}{c_+^2k^2}}\right)\end{aligned}$$

Interface behavior:

$$\frac{\partial\hat{\eta}}{\partial t} = \mathcal{L}^{-1}\left[\frac{(r-1)V\eta_0ks^3}{\Omega(s)}\right].$$

Short-time behavior

$$\frac{\partial \hat{\eta}}{\partial t} = \mathcal{L}^{-1}\left[\frac{(r-1)V\eta_0 k s^3}{\Omega(s)}\right].$$

Inviscid fluid: classic impulsive RM result recovered when $c_{\pm} = 0$

$$\frac{\partial \hat{\eta}}{\partial t} = \frac{(r-1)}{(r+1)} V\,\eta_0\,k = A\,V\,\eta_0\,k$$

This is also the elastic-solid initial growth rate, since when s is large

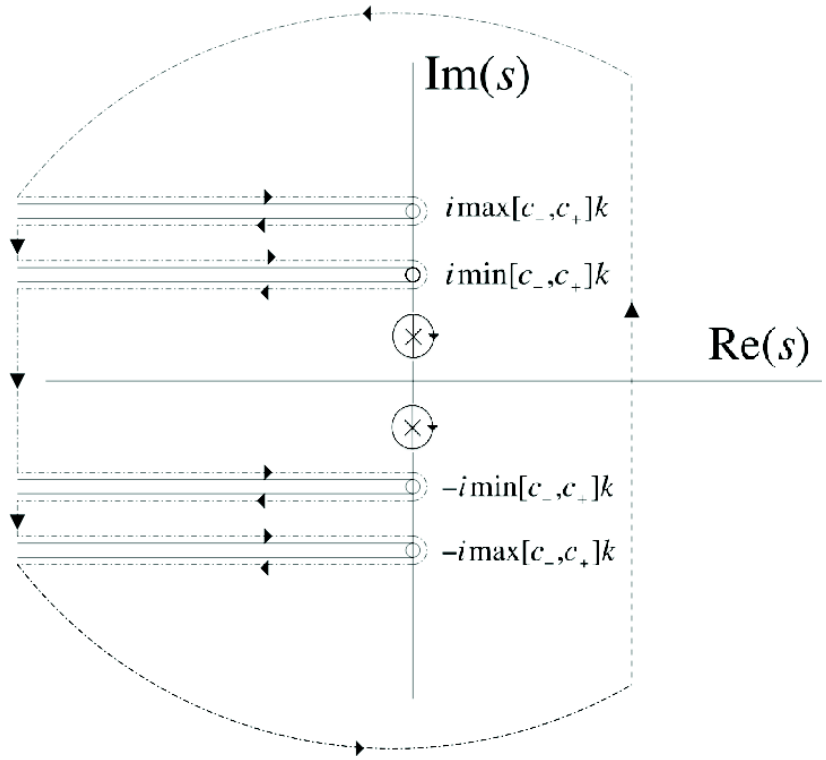
$$U_1(0,s) \sim A\,V\eta_0\,k\,\frac{1}{s}, \qquad s \to \infty (t \to 0)$$

For $t << \dfrac{1}{k\,c_{\pm}}$ vorticity is located near the interface and so interface behaves as for inviscid fluid

Long-time behavior

$$\frac{\partial \hat{\eta}}{\partial t} = \mathcal{L}^{-1}\left[\frac{(r-1)V\eta_0 k s^3}{\Omega(s)}\right].$$

- $\Omega(s)$: 4-sheet Riemann surface
- Generally 4 finite branch points
- Branch constructed with $\sqrt{}$ having positive real part, $Re(s) > 0$
- Branch cuts chosen for convenience
- Find poles of $\Omega(s)$ on constructed branch
- Pole locations are functions of $r, \frac{c_-}{c_+}$
- Contributions to Bromwich integral from both poles and branch cuts



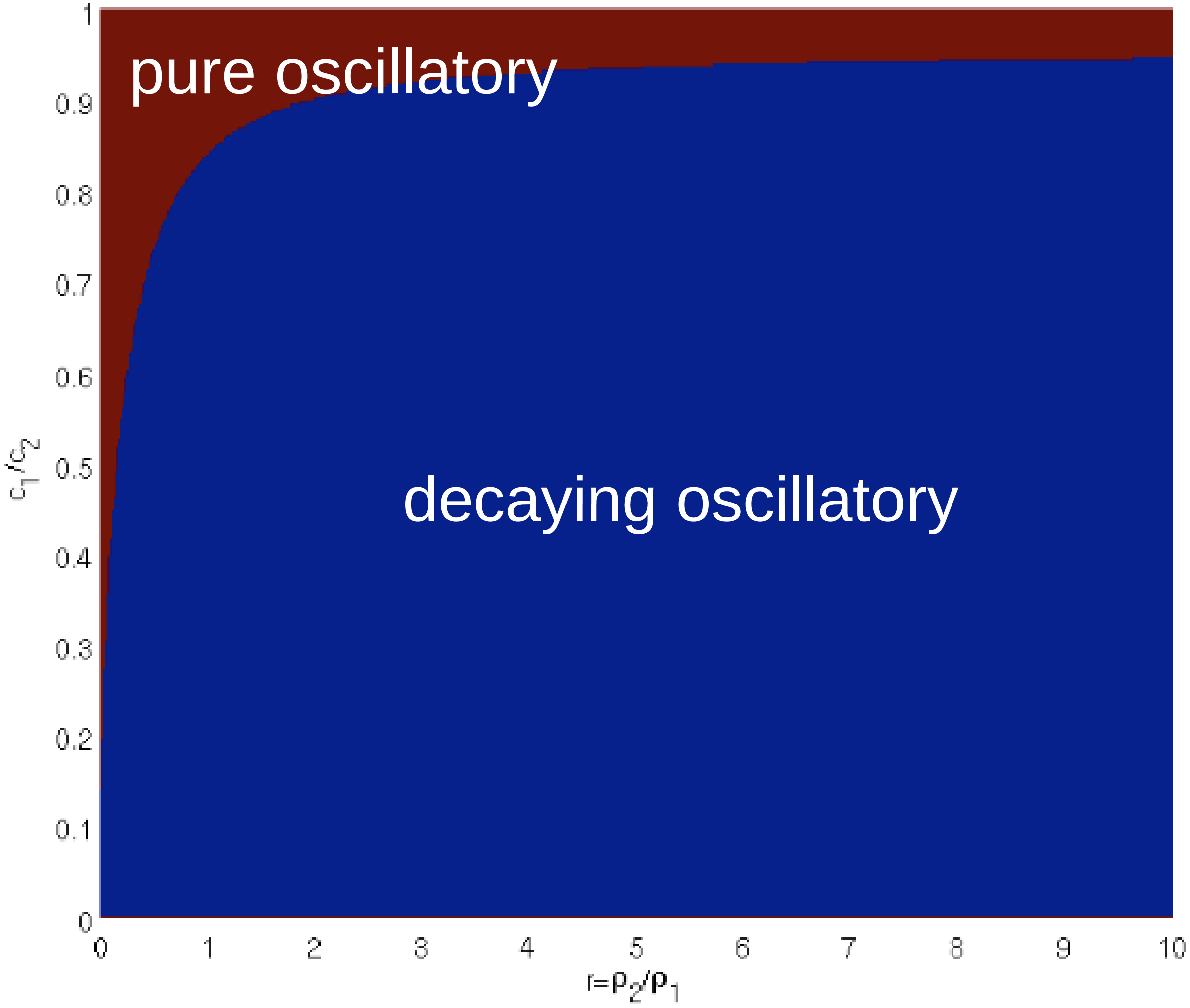
Long-time behavior

- All poles are simple and lie in $Re(s) \leq 0$
- As material properties $r, \frac{c_-}{c_+}$ change, poles move along $Im(s)$ axis and then either vanish or move off axis
- Branch cuts lead to algebraic decay:

$$\hat{\eta}(t) \sim \left(t^{-3/2}, t^{-5/2}, t^{-5/2} \ldots\right) e^{\pm i\, c_\pm\, t}$$

- Poles/branch cut analysis shows that interface is always stable
- Two types of behavior depending on $r, \frac{c_-}{c_+}$
  - (a) pure oscillatory: when $\frac{c_-}{c_+}$ close to unity (pole on $Im(s)$)
  - (b) Decaying oscillatory behavior (branch cuts)

Stability of the interface



Stability of the interface

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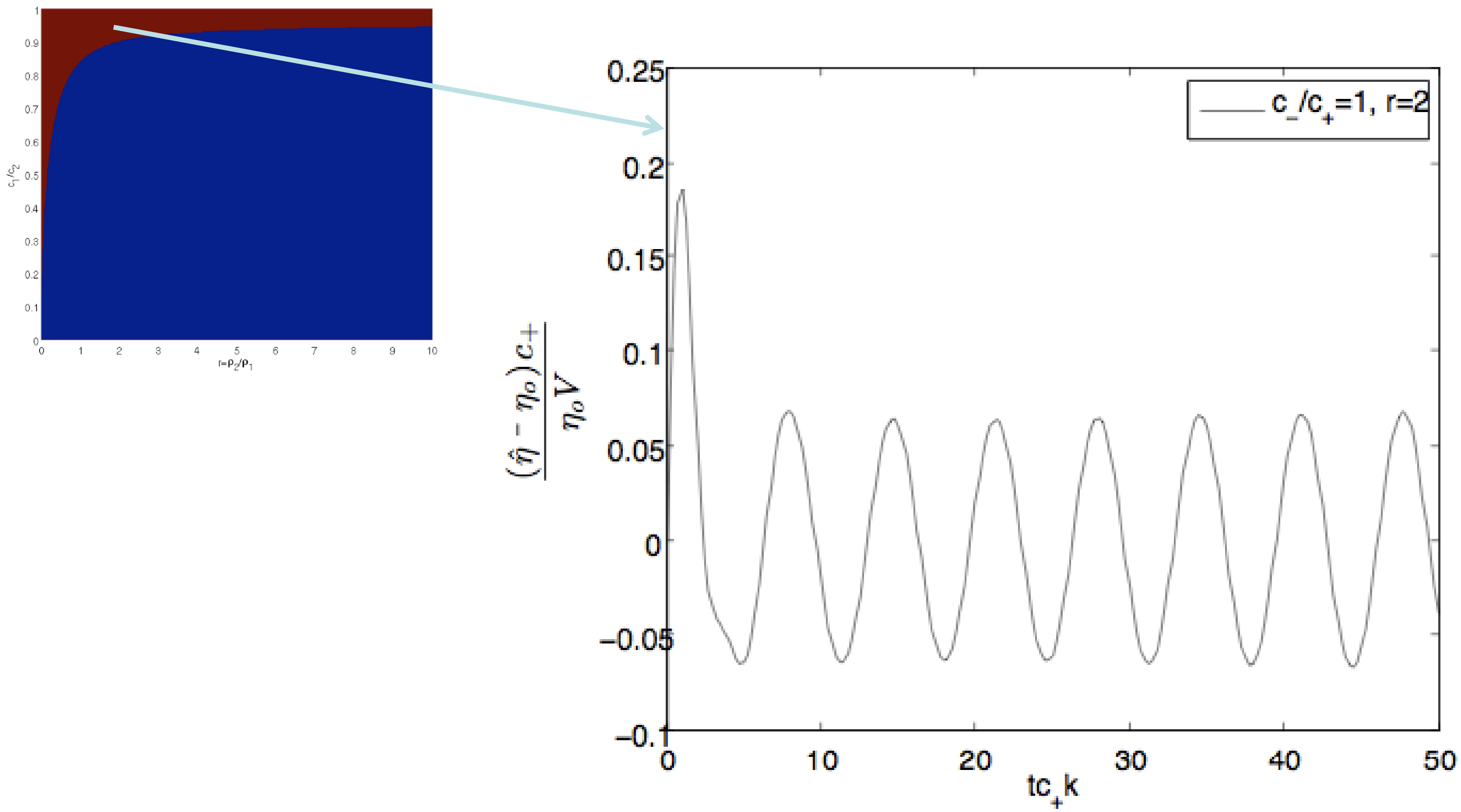
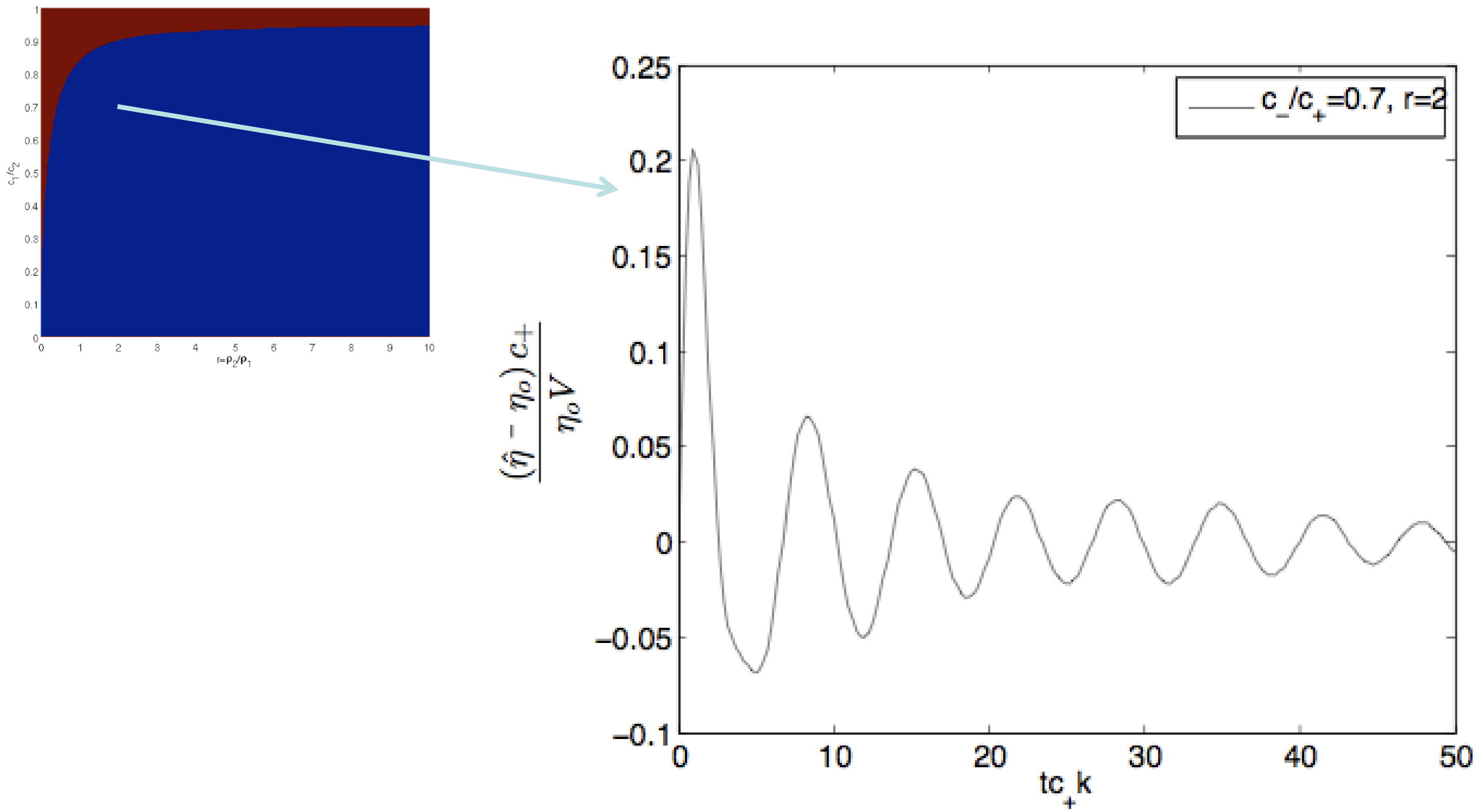
pure oscillatory



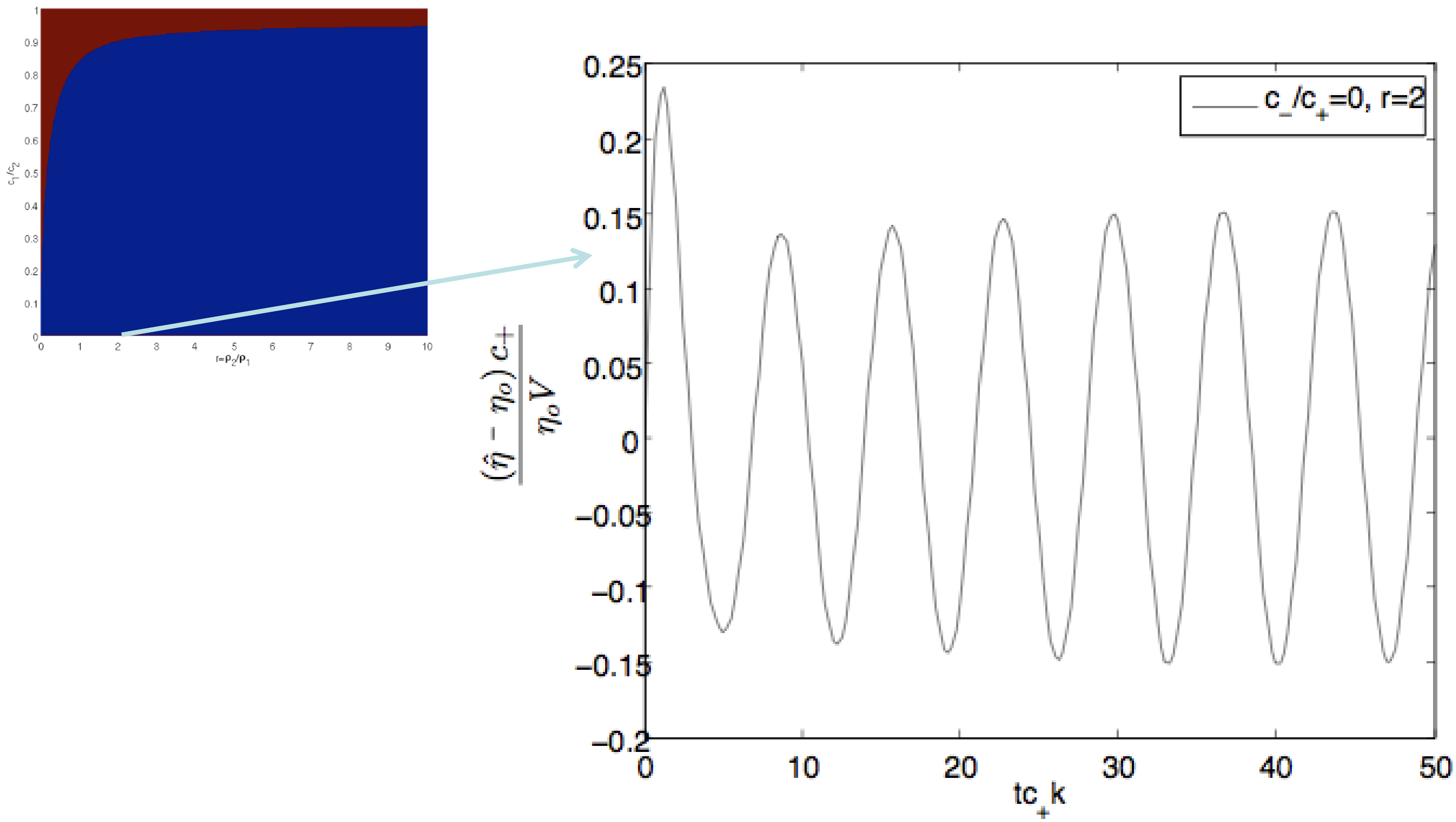
Stability of the interface

decaying oscillatory



Stability of the interface

Asymptotic oscillatory



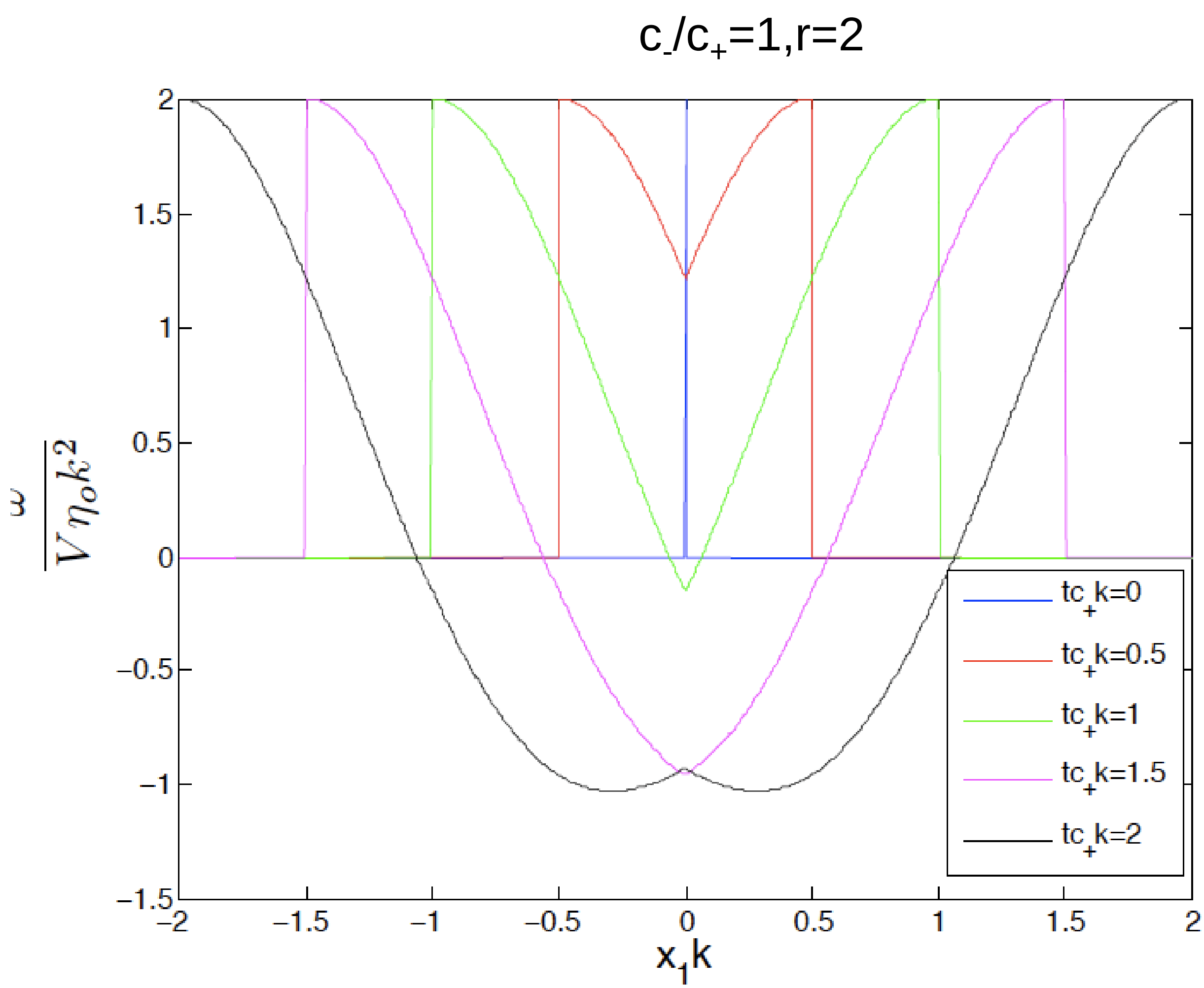
Why stable?

- We can look at the vorticity equation:

$$\begin{aligned}
\frac{\partial^2 \omega'}{\partial t^2} &= c^2 \frac{\partial^2 \omega'}{\partial x_i^2} \\
\omega'(x_1, x_2, t) &= \omega(x_1, t)\, e^{ikx_2} = i\left(-k\hat{u}_1 + \frac{1}{k}\frac{\partial^2 \hat{u}_1}{\partial x_1^2}\right) e^{ikx_2} \\
\hat{\omega}_\pm(x_1, s) &= -2ikU_1^\pm(s,0)\, e^{\mp\sqrt{\frac{s^2}{c_\pm^2 k^2}+1}\; kx_1} \\
\omega_\pm(x_1, t) &= \int_0^t \omega(x_1 = 0, \tau)\, \mathcal{L}^{-1}[\exp(\mp\sqrt{\frac{s^2}{c_\pm^2 k^2} + 1}\; kx_1)]\; d\tau
\end{aligned}$$

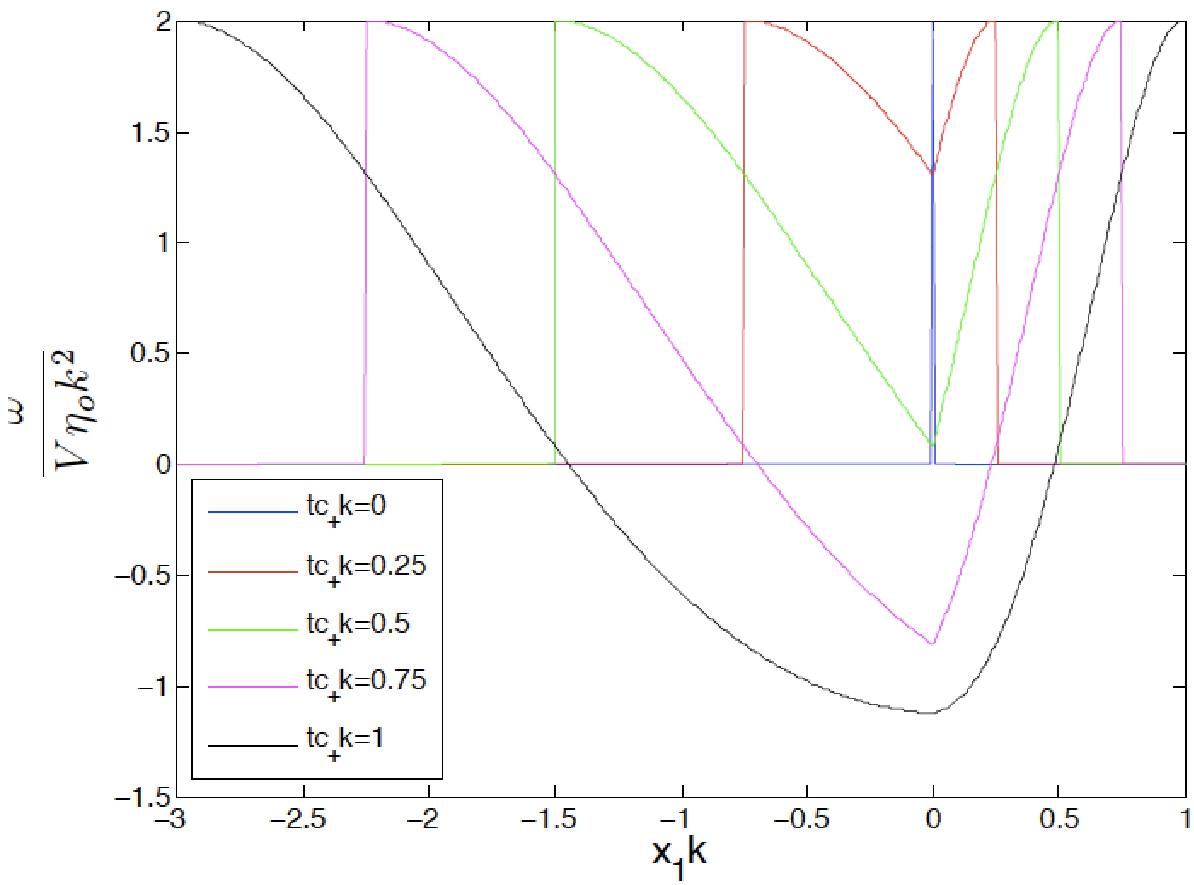
- Initial vorticity created by impulsive acceleration is carried away from the interface by shear waves.
- So vorticity first created near interface is carried away and distributed between shear waves and the interface

Vorticity distribution

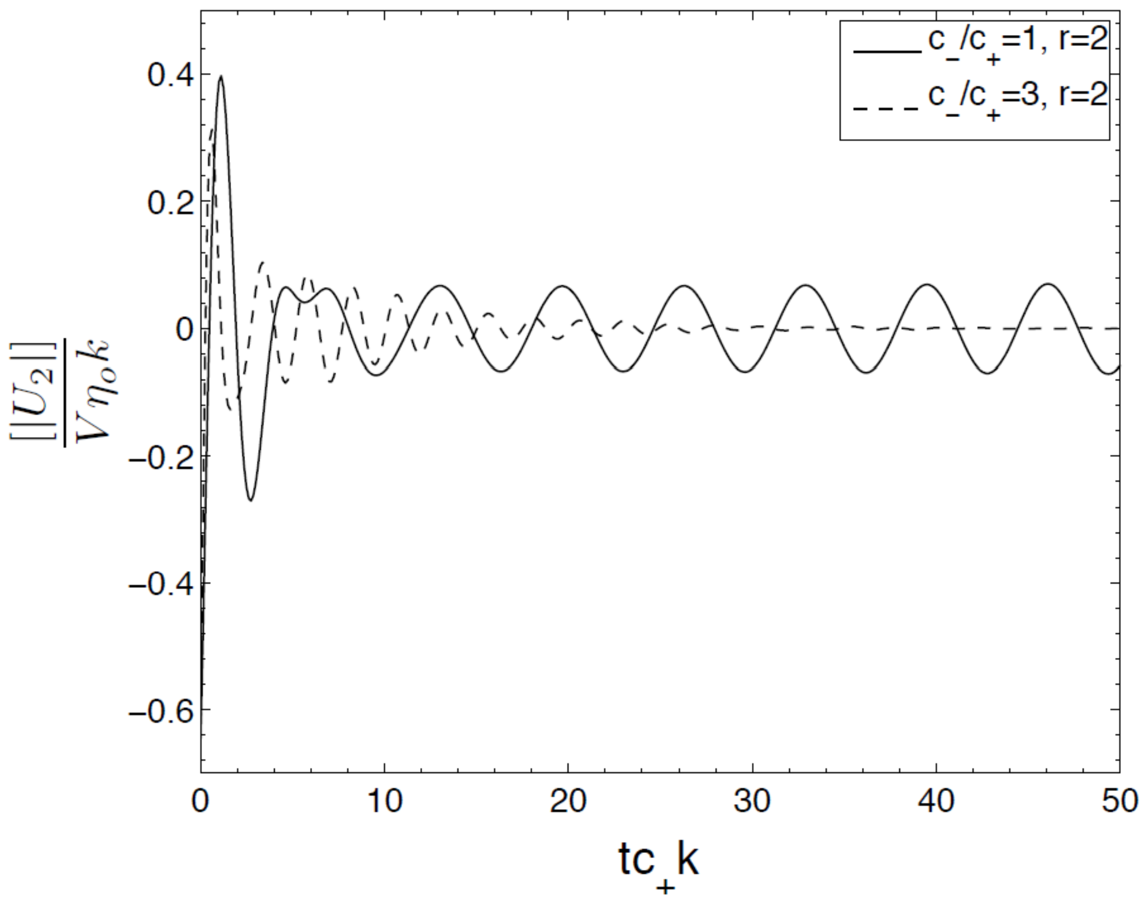


Vorticity distribution

$c_-/c_+=3, r=2$



Tangential velocity jump



Evolution of velocity jump across interface; two cases

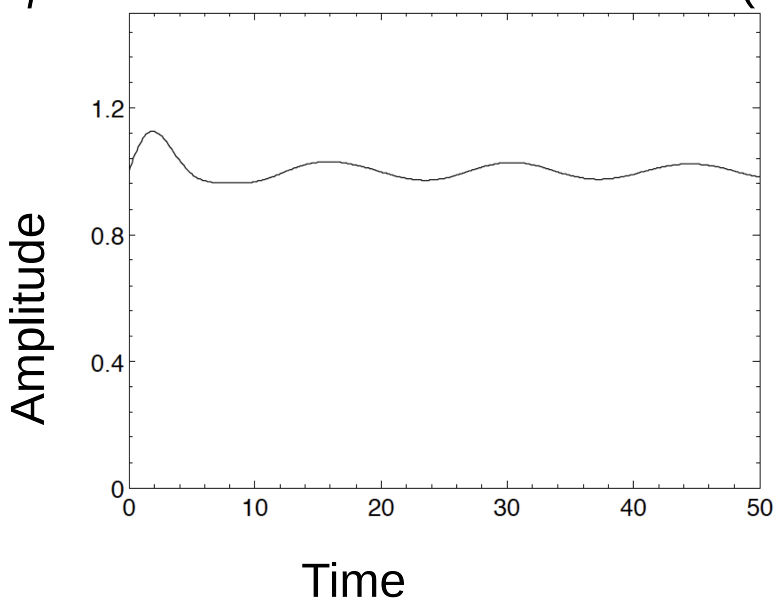
Comparison with other studies

- Y.Plohr and B.Plohr, JFM, 2005
  - Linearized shock problem:
    - base problem+perturbation
  - Compressible
  - Numerical solution of linearized PDEs

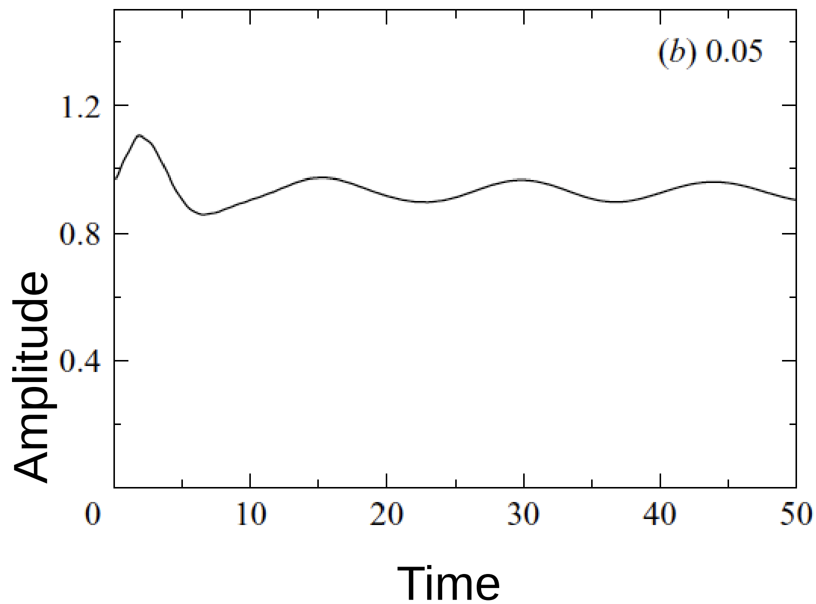
$c_-/c_+=1.56, r=6.16$

(Al/Ta system)

Impulsive model



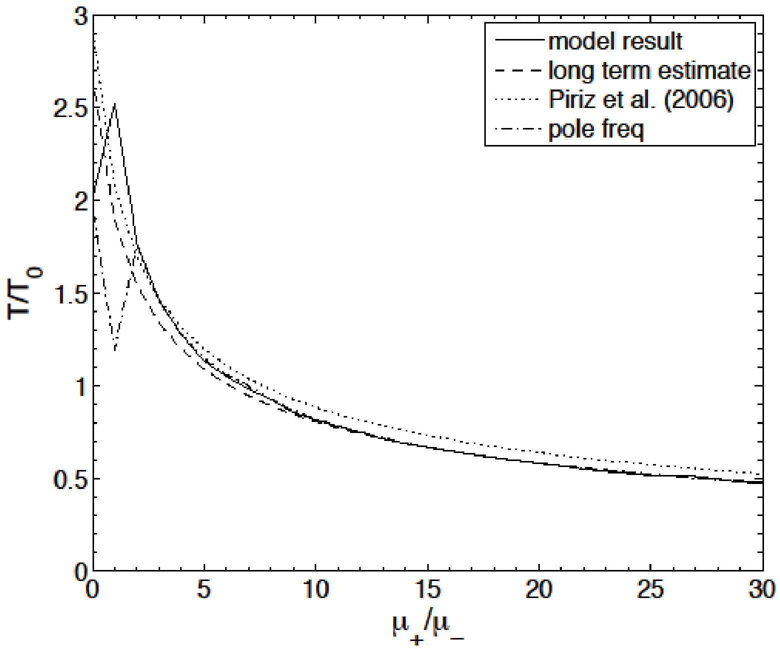
Plohr & Plohr



Comparison with other studies

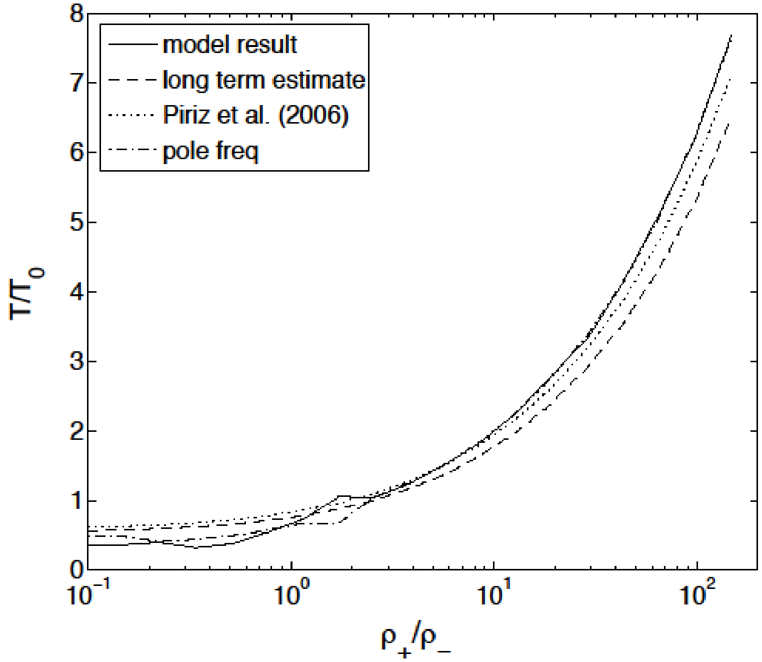
- Present solution
  - Long-time asymptotics available
  - In oscillatory regime, oscillation period is

$$\frac{T}{T_0} = \sqrt{\frac{1+r}{1+\frac{\mu_+}{\mu_-}}}, \quad T_0 = \frac{2\pi}{k}\sqrt{\frac{\rho_-}{\mu_-}}$$



- Piriz et Al., PRE, 2006
  - Local analysis and simulation
  - Predicts pure oscillatory behavior only

$$\frac{T}{T_0} = \frac{1.55}{\sqrt{2}}\sqrt{\frac{1+r}{1+\frac{\mu_+}{\mu_-}}}$$



Conclusions

- Impulsive Richtmyer-Meshkov flow for incompressible elastic solids is stable.
- Agrees with previous semi-analytical studies (Plohr and Plohr 2005, and Piriz et. Al 2006) over their range of validity.
- Two distinct types of interface behavior are found:
  - Pure oscillatory behavior when shear speeds in the materials are approximately matched and in the case when one material has no shear strength
  - Decaying oscillation when the shear speeds are sufficiently different.
- Stability is achieved by the action of shear waves, which carry vorticity away from the interface
- Analysis predicts other aspects of interface behavior including oscillatory period as a function of material properties.